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A SET OF AXIOMS FOR DIFFERENTIAL GEOMETRY

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1. *Introductory.*—The axioms set forth in this note are intended to describe the class of manifolds of n dimensions to which the theories nowadays grouped together under the heading of differential geometry are applicable. The manifolds are classes of elements called points, having a structure which is characterized by means of coördinate systems. A *coördinate system* is a (1–1) correspondence, $P \longrightarrow x$, between a set of points, $[P]$, of the manifold, and a set, $[x]$, of ordered sets of n real numbers, $x = (x^1, \ldots, x^n)$. For convenience we call any ordered set of n real numbers an arithmetic point and the totality of arithmetic points, for a fixed n, the arithmetic space of n dimensions. Each point P which corresponds in a coördinate system, $P \longrightarrow x$, to an arithmetic point x, is said to be represented by x. The set, $[P]$, of all points represented in a given coördinate system is called the *domain* of the coördinate system, and the set, $[x]$, of the arithmetic points which represent them is called its *arithmetic domain*.

If $P \longrightarrow x$ and $P \longrightarrow y$ are two coördinate systems having the same domain, the transformation $x \longrightarrow y$ of the arithmetic space, which is the resultant of $x \longrightarrow P$ followed by $P \longrightarrow y$, is called the transformation of coördinates from $P \longrightarrow x$ to $P \longrightarrow y$, or the transformation between $P \longrightarrow x$ and $P \longrightarrow y$. The general scheme of the axioms is to characterize a class of "allowable coördinate systems" by means of the analytic properties of the transformations between them. In order to state these properties clearly we recall a few arithmetic theorems and definitions:

A set of arithmetic points given by

$$| x^i - x_0^i | < \delta,$$

for some positive δ, will be called a *box*, and the point x_0 will be called its center. A set of points $[x]$ will be called a region if each x is the center of a box which is contained in $[x]$.

A function $F(x^1, \ldots, x^n)$, defined over a region, $[x]$, will be described

as of class u if it and its first u derivatives exist and are continuous at each point of $[x]$. Here u can be any positive integer. A function will be described as belonging to the class ∞ if all its derivatives exist. Continuous functions will be described as belonging to the class 0, and analytic functions, i.e., functions which can be expanded in a power series about each point in $[x]$, will be described as of class ω. We deal with single valued functions only.

Let $y^1(x), \ldots, y^n(x)$ be n functions of class $u > 0$, defined over a region $[x]$. The equations

$$y^i = y^i(x) \qquad (1.1)$$

define a transformation, $x \longrightarrow y$, of $[x]$ into a set of points $[y]$. If the Jacobian

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)}$$

does not vanish in $[x]$ it follows from the implicit function theorem that $[y]$ is also a region. If the transformation $x \longrightarrow y$ is non-singular, i.e., has a single-valued inverse, it will be called a *regular transformation of class u*. Unless otherwise stated it is to be assumed that any transformation to which we refer is of this type. We assume u to be fixed (either as 0, 1, $\ldots$, ∞ or ω), and shall often omit the words "of class u," as applied to functions, transformations and, later on, to n-cells.

A regular transformation, $x \longrightarrow y$, of a region $[x]$ into a region $[y]$, followed by a regular transformation, $y \longrightarrow z$, of $[y]$ into $[z]$ is a regular transformation, $x \longrightarrow z$, of $[x]$ into $[z]$. Moreover, the inverse of a regular transformation is regular.

The last two sentences assert that the totality of regular transformations between regions is what we call a *pseudo-group*. A pseudo-group is any set of transformations which satisfy the conditions:

(1). *If the resultant of any two exists, it is in the set.*

(2). *The inverse of each transformation in the set is also in the set.*

The pseudo-group of regular transformation between regions will be called the *pseudo-group of class u*.

An *arithmetic n-cell of class u* is any set of arithmetic points obtained from a box by a regular transformation of class u. Thus the arithmetic space is itself an n-cell. Moreover, any two n-cells of class u are equivalent under the pseudo-group of class u. If $u \leqq u'$, any function, transformation or n-cell of class u' is also a function, transformation or n-cell of class u.

2. *The First Group of Axioms.*—The axioms are arranged in three groups, the first being:

A_1. *The transformation between two allowable coördinate systems which have the same domain is regular if the arithmetic domain of one of them is a region.*

A_2. *Any coördinate system obtained by a regular transformation of coordinates from an allowable coördinate system is allowable.*

Definition: The image in an allowable coordinate system of a box will be called an *n-cell of class u.*

A_3. *The correspondence in which each point of an n-cell corresponds to its image in an allowable coördinate system is an allowable coördinate system.*

3. *The Second Group of Axioms.*—Let $[K_\alpha]$ be any set of allowable coordinate systems, finite or infinite. Let K be the correspondence in which each point in the domain of at least one K_α corresponds to every arithmetic point by which it is represented in at least one K_α. If the correspondence K is (1–1) it will be called the *union* of $[K_\alpha]$. It can be shown that the union of $[K_\alpha]$ exists if the union of each pair K_α and K_β exists.

The axioms of the second group are:

B_1. *Any coördinate system which is the union of a set of allowable coordinate systems whose domains are n-cells is allowable.*

B_2. *Each allowable coördinate system is the union of a set of allowable coördinate systems whose domains are n-cells.*

4. *The Third Group of Axioms.*—The axioms of the third group are:

C_1. *If two n-cells have a point in common they have in common an n-cell containing this point.*

C_2. *If P and Q are any two distinct points there is an n-cell C_P, containing P, and an n-cell C_Q, containing Q, such that C_P and C_Q have no point in common.*

C_3. *There are at least two points.*

5. *Regular Manifolds.*—Any space satisfying the axioms A, B and C will be called an *n-dimensional manifold of class u*, or a *regular manifold.* When $u = 0$ the theory of a regular manifold is a branch of analysis situs. When $u = 1$ it is possible to define a tangent space of differentials at each point of the manifold, and in addition to pure continuity considerations we can apply some of the formal machinery of the differential calculus. When $u = 2$ we have second differentials, affine connections and so on.

It would obviously be impossible in the space here available to show how differential geometry is built out of these axioms. We have tried to do this in a small book called "Foundations of Different Geometry," which we hope to publish as a Cambridge Tract, and will not deal further with it here. Instead, we make a few remarks about the special peculiarities of our axioms, and prove their independence.

6. *Topological Considerations.*—From the axioms C it follows that a regular manifold satisfies the axioms given by F. Hausdorff (Mengenlehre, Leipzig, 1914, p. 213) for a topological space. Indeed, if C be taken as a separate set of axioms with points and n-cells as the undefined elements,

the spaces which they determine satisfy Hausdorff's axioms, provided each n-cell is taken as a neighborhood of each point in it. In the presence of the axioms A and B, a regular manifold is what some writers call a homogeneous topological space of n dimensions, that is, a topological space, each of whose neighborhoods is homeomorphic with the arithmetic space of n dimensions. This statement is equivalent to the theorem:

An allowable coördinate system is a homeomorphism between its domain and its arithmetic domain.

Let $[x]$ be a set of arithmetic points having a limit point x', where $[x]$ and x' are both in the arithmetic domain of an allowable coördinate system $P \longrightarrow x$. Let $[P]$ and P' be the respective images of $[x]$ and x'. It will follow that P' is a limit point of $[P]$, if every n-cell C contained in the domain of $P \longrightarrow x$, and containing P', contains points of $[P]$ other than P'. From the axioms A it follows that the image of C in $P \longrightarrow x$ is an arithmetic n-cell containing x', and since x' is a limit point of $[x]$, a point of $[x]$ other than x'. Therefore C contains points of $[P]$ other than P', and P' is a limit point of $[P]$. Therefore the transformation $x \longrightarrow P$ is continuous.

Interchanging the parts played by the regular manifold and the arithmetic space, it follows by a similar argument that the transformation $P \longrightarrow x$ is continuous. It is, therefore, a homeomorphism.

According to the theorem just proved, the domain of any allowable coördinate system is topologically equivalent to some arithmetic region. No arithmetic region is self-compact (a region U is said to be self-compact if any infinite set of points in U has at least one limit point in U) and we have as a corollary: *The domain of an allowable coördinate system is not self-compact.* Therefore a manifold such as the surface of a sphere or an anchor ring cannot be represented completely in a single allowable coordinate system.

7. *Consistency and Independence Examples.*—As a consistency example take the arithmetic space of n dimensions with regular transformations between regions for allowable coördinate systems. The axioms A, B and C are obviously satisfied by this set of coördinate systems. We shall show that the axioms are independent by giving consistency examples for the sets obtained by denying each one in turn.

Denying A_1. Take the arithmetic space with all non-singular transformations which operate on regions for allowable coördinate systems. Then the domain of an allowable coördinate system is a region, but the arithmetic domain may be any set of points with the power of the continuum.

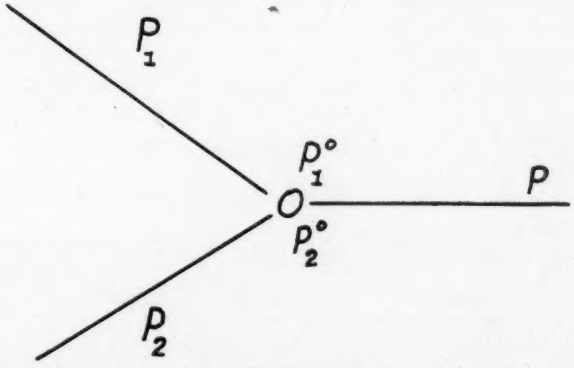
Denying A_2. Take the arithmetic space with any correspondence in which each point of a region corresponds to itself, and only such correspondences, as allowable coördinate systems.

Denying A_3. Limit the allowable coördinate systems in the arithmetic space to those regular transformations in which the whole space corresponds to the whole or a part of itself.

Denying B_1. Limit the allowable coördinate systems in the arithmetic space to regular correspondences between n-cells.

Denying B_2. The allowable coördinate systems are to be those given in the consistency proof together with all those in which a single point corresponds to a single point.

Denying C_1. The space ($n = 1$) is to be the arithmetic space of two dimensions, and an allowable coördinate system is to be a regular correspondence between a set (finite or infinite) of 1-cells in this space, no two of which have a common point, and a set of segments in the arithmetic



space of one dimension. Any such correspondence is to be an allowable coördinate system. Thus the correspondence given by

$$x = t,\ y = 0,$$

for $-1 < t < 1$, will be one allowable coördinate system and that given by

$$x = 0,\ y = t,$$

for $-1 < t < 1$, will be another. The domains of these two coördinate systems will have the origin (0, 0) in common, and no other point. Therefore C_1 is not satisfied.

Denying C_2. Let $[P]$ be a set of points in a (1–1) correspondence, K, with the interval $0 < t < 1$. Let $[P_1]$ and $[P_2]$ be sets of points in (1–1) correspondences K_1 and K_2, respectively, with the real numbers which are not greater than zero. The space is to consist of these three sets of points with the allowable coördinate systems defined as follows: The correspondence K'_1 in which each P corresponds to its image in K, and each P_1 corresponds to its image in K_1, is to be an allowable coördinate

system. The correspondences obtained from K'_1 according to the axioms A are also to be allowable coördinate systems. The analogous correspondences for the set of points consisting of $[P]$ together with $[P_2]$ are to be allowable coördinate systems, and the remaining allowable coördinate systems are the unions of these.

Let P_1^0 and P_2^0 be the points which correspond to zero under K_1 and K_2, respectively. Any two n-cells containing P_1^0 and P_2^0 have a point in common, contradicting C_2.

Denying C_3. The space shall have no points.

8. *Other Pseudo-groups.*—Let g be any pseudo-group of continuous transformations between regions in the arithmetic space, such that the axioms A and B are consistent when the space is the arithmetic space and the transformations of g are taken as allowable coördinate systems. We obtain a consistent set of axioms if we substitute "transformation of g" for "regular transformation" in the axioms A and leave the axioms B and C unchanged. The spaces which satisfy these axioms constitute a sub-class of the manifolds of class 0. It is a special case of this remark that the manifolds of class 0 include those of class 1, which include those of class 2, and so on.

Let us consider a few examples of other pseudo-groups which give rise to such spaces.

(1). Let g be the pseudo group of regular transformations $x \longrightarrow y$ with a constant Jacobian $\left|\frac{\partial y}{\partial x}\right|$. In a space determined by g there is a definition of ratios of volume. This pseudo-group has an invariant sub-pseudo-group, g', of transformations whose Jacobian is unity. In the spaces g' there will be an invariant unit of volume. The relation between the spaces g and the spaces g' is analogous to that between affine and equiaffine spaces.

(2). Let g^+ be the pseudo-group of direct transformations of class u, that is, regular transformations with a positive Jacobian. The spaces g^+ are oriented manifolds of class u.

(3). Let g be the pseudo-group of linear transformations between regions. This may be called the affine pseudo-group, as apart from the affine group whose transformations carry the arithmetic space into itself as a whole. The spaces defined by the affine pseudo-group may be called locally flat affine spaces.

(4). Let $n = 2r$, and let g be the pseudo-group of regular contact transformations. That is to say, any transformation of g is given by equations of the form

$$\begin{cases} \bar{x}^\lambda = \bar{x}^\lambda (x^1, \ldots, x^r, p_1, \ldots, p_r), \\ \bar{p}_\lambda = \bar{p}_\lambda (x^1, \ldots, x^r, p_1, \ldots, p_r), \ (\lambda = 1, \ldots, r) \end{cases}$$

where $\bar{x}^\lambda(x, p)$ and $\bar{p}_\lambda(x, p)$ are functions of class u such that

$$\bar{p}_\lambda d\bar{x}_\lambda = p_\lambda dx^\lambda + dW(x, p),$$

$W(x, p)$ being an arbitrary function of class u.

(5). Let $n = 2r$, and let g be the pseudo-group of transformations given by

$$w^\lambda = w^\lambda(z^1, \ldots, z^r), (\lambda = 1, \ldots, r),$$

where z and w are complex variables, and $w^\lambda(z)$ are analytic functions. If

$$z^\lambda = x^\lambda + iy^\lambda,$$
$$w^\lambda = u^\lambda + iv^\lambda,$$

the transformations of g may also be written

$$u^\lambda = u^\lambda(x, y),$$
$$v^\lambda = v^\lambda(x, y).$$

If $n = 2$ the spaces g are Riemann surfaces (cf. H. Weyl, "Die Idee der Riemannschen Flache," Leipzig, 1913, p. 36) and their geometry is two dimensional conformal geometry.

In a 2-dimensional manifold of class ω, a scalar, $f(P)$, of highest class has as its component in any allowable coördinate system an analytic function of two real variables. But in a space g it has a harmonic function as its component, i.e., a function which is the real or imaginary part of an analytic function of a complex variable. That is to say, we can define complex point-functions, $f(P)$, such that

$$F(x, y) = f(P),$$

is an analytic function of $x + iy$, where $P \longrightarrow (x, y)$ is any one of the class of coördinate systems which satisfies the axioms g.

In general any pseudo-group, G, of continuous transformations between regions in the arithmetic n-space determines a unique pseudo-group which defines a special class of regular manifolds. For g is that pseudo-group which contains G and satisfies the axioms A_3, B_1 and B_2, when the transformations of g are taken as allowable coördinate systems.

9. *Two Existence Theorems.*—The use of the pseudo-group of transformations of coördinates of class $<\omega$ raises a question as to the status of operations which are ordinarily described by means of power series. A case in point is the transformation to normal coördinates in the affine geometry of paths. The space of this geometry is a regular manifold with a symmetric affine connection defined at every point. In any allowable coördinate system the paths are given by the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{9.1}$$

where Γ^i_{jk} are the components of the affine connection. The transformation law of an affine connection involves the second but not the higher derivatives, and therefore it is an invariant condition to require Γ^i_{jk} to be functions of class u' if $u' \leqq u - 2$, but not if $u' > u - 2$.

If $u \geqq 3$ and if Γ^i_{jk} are of class $u - 2$ at a point Q, there exists an n-cell C_Q, containing Q, which is represented in a coördinate system $P \longrightarrow y$, in which the paths through Q are given by linear equations of the form

$$y^i = p^i s. \tag{9.2}$$

The transformation of coördinates between $P \longrightarrow y$ and any allowable coördinate system in which Q is represented, is of class $u - 2$.

The equations (9.1) admit a unique set of solutions (see, for instance, L. Bieberbach, "Theorie der Differentialgleichungen," Berlin, 1926, pp. 115–116),

$$f^i(q, p, s),$$

which satisfy the initial conditions

$$\begin{cases} f^i(q, p, 0) = q^i, \\ \left(\dfrac{\partial f^i}{\partial s}\right)_{s=0} = p_i, \end{cases}$$

where $Q \longrightarrow q$ in a given allowable coordinate system, $P \longrightarrow x$. If λ is any constant it follows from the form of the equations (9.1) that

$$f^i(q, p, \lambda s)$$

are the solutions which satisfy the initial conditions

$$\begin{cases} f^i(q, p, 0) = q^i, \\ \left(\dfrac{\partial f^i}{\partial s}\right)_{s=0} = \lambda p^i. \end{cases}$$

That is to say,

$$f^i(q, p, \lambda s) = f^i(q, \lambda p, s).$$

and the solutions may therefore be written

$$F^i(q, y),$$

where

$$y^i = p^i s$$

It also follows that $F^i(q, y)$ are of class $u - 2$ for values of y in some arithmetic n-cell containing the origin (see L. Bieberbach, loc. cit., pp. 39–41).

Since

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 \quad p^j = p^i$$

for all values of p, we have

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta^i_j, \tag{9.3}$$

and therefore the Jacobian $\left|\frac{\partial x}{\partial y}\right|$ does not vanish for values of y near the origin. Therefore the equations

$$x^i = F^i(q, y) \tag{9.4}$$

can be solved to give a regular transformation $x \longrightarrow y$ of class $u - 2$, which carries some arithmetic n-cell contained in the arithmetic domain of $P \longrightarrow x$, and containing q, into an arithmetic n-cell containing the origin. Therefore, $x \longrightarrow y$ is a transformation between $P \longrightarrow x$ and a coördinate system, $P \longrightarrow y$, satisfying the required conditions. Because $P \longrightarrow y$ also satisfies the conditions (9.3) it is called a normal coördinate system at the point Q, for the coördinate system $P \longrightarrow x$. Clearly the union of all normal coördinate systems at Q, for the same coördinate system, $P \longrightarrow x$, exists, and may be called *the* normal coördinate system at Q for $P \longrightarrow x$. Normal coördinate systems are not necessarily allowable except when $u = \infty$, and when $u = \omega$.

10. When the curvature tensor or one of its generalizations arises as an integrability condition, an existence theorem is in the background which, for u arbitrary, takes the following form:

Let

$$F^\alpha(x^1, \ldots, x^n, z^1, \ldots, z^m)$$

(Greek indices will run from 1 to m, Roman indices from 1 to n) *be functions of class 1 in the region*

$$| x^i | < 1, | z^\alpha | < 1. \tag{10.1}$$

If, and only if,

$$\frac{\partial F^\alpha_i}{\partial x^j} - \frac{\partial F^\alpha_j}{\partial x^i} + \frac{\partial F^\alpha_i}{\partial z^\beta} F^\beta_j - \frac{\partial F^\alpha_j}{\partial z^\beta} F^\beta_i = 0, \tag{10.2}$$

the differential equations

$$\frac{\partial z^\alpha}{\partial x^i} = F^\alpha_i(x, z), \tag{10.3}$$

admit a unique set of solutions

$$z^\alpha(x_0, z_0, x),$$

which satisfy given initial conditions

$$z^\alpha(x_0, z_0, x_0) = z_o^\alpha, \tag{10.4}$$

where x_0 and z_0 are any values of the variables x and z which satisfy (10.1).

We give a short proof of this theorem, which is perhaps well known, because we have not seen it proved elsewhere except when the functions F are analytic.

By a suitable transformation we can reduce the problem to the case where $x_0 = z_0 = 0$, and we suppose this to have been done.

Instead of (10.3), let us consider the ordinary differential equations

$$\frac{dz^\alpha}{dt} = p^i F_i^\alpha(p^1 t, \ldots, p^n t, z^1, \ldots, z^m), \tag{10.5}$$

involving the parameters $p^1, \ldots, p^n$. These equations admit a unique set of solutions

$$z^1(p, t), \ldots, z^m(p, t),$$

all of which vanish with t. By substituting λt for t it follows from the form of (10.5) that

$$z^\alpha(p, \lambda t) = z^\alpha(\lambda p, t),$$

or that the solutions may be written

$$z^1(x), \ldots, z^m(x), \tag{10.6}$$

where

$$x^i = p^i t.$$

Moreover, $z^\alpha(x)$ are of class 1 in some n-cell containing the origin (Bieberbach, loc. cit., pp. 39–41).

We remark in passing that (10.3) have at most one solution satisfying the given initial conditions. For if $u(x)$ and $v(x)$ are any solutions, $u(pt)$ and $v(pt)$ both satisfy (10.5), and vanish with t. But (10.5) only admits one such solution for given values of $p^1, \ldots, p^n$. Therefore $u(x) = v(x)$.

We have now to show that if the conditions (10.2) are satisfied, the functions (10.6) satisfy (10.3). We have

$$z^\alpha = \int_0^t p^i F_i^\alpha(ps, z(ps))ds.$$

When $x \neq 0$, therefore,

$$\frac{\partial z^\alpha}{\partial x^j} = \frac{1}{t}\frac{\partial z^\alpha}{\partial p^j}$$

$$= \frac{1}{t}\int_0^t \left\{\left(\frac{\partial F_i^\alpha}{\partial x^j} + \frac{\partial F_i^\alpha}{\partial z^\beta}\frac{\partial z^\beta}{\partial x^j}\right)p^i s + F_j^\alpha\right\} ds,$$

$$= F_j^\alpha + \frac{1}{t}\int_0^t s\left(\frac{\partial F_i^\alpha}{\partial x^j} - \frac{\partial F_j^\alpha}{\partial x^i} + \frac{\partial F_i^\alpha}{\partial z^\beta}\frac{\partial z^\beta}{\partial x^j}\right.$$

$$\left. - \frac{\partial F_j^\alpha}{\partial z^\beta} F_i^\beta\right)p^i\, ds, \qquad (10.7)$$

where we have integrated by parts and used the identities

$$\left(\frac{\partial z^\alpha}{\partial x^i} - F_i^\alpha\right)p^i = 0,$$

to obtain the last equality. From these identities it also follows that (10.3) are satisfied for $x = 0$. From (10.2) and (10.7) we have

$$t\left(\frac{\partial z^\alpha}{\partial x^j} - F_j^\alpha\right) = \int_0^t p^i \frac{\partial F_i^\alpha}{\partial z^\beta}\, s\left(\frac{\partial z^\beta}{\partial x^j} - F_j^\beta\right) ds.$$

Therefore

$$t\left(\frac{\partial z^1}{\partial x^j} - F_j^1\right), \ \ldots, \ t\left(\frac{\partial z^m}{\partial x^j} - F_j^m\right)$$

satisfy the differential equations

$$\frac{dy^\alpha}{dt} = a_\beta^\alpha(t)\, y^\beta, \qquad (10.8)$$

where $$a_\beta^\alpha(t) = p^i \frac{\partial F^\alpha(pt, z(pt))}{\partial z^\beta}.$$

These functions are continuous and therefore there is only one solution which vanishes with t, namely, $(0, \ldots, 0)$. Therefore

$$\frac{\partial z^\alpha}{\partial x^j} = F_j^\alpha(x, z),$$

and so (10.2) are sufficient conditions for (10.3) to be completely integrable. They are obviously necessary, and so the theorem is established.

SOLUTIONS OF A CERTAIN PARTIAL DIFFERENTIAL EQUATION

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1. The partial differential equation

$$\frac{\partial u}{\partial t} = x\left(\frac{\partial^2 u}{\partial x^2} - u\right) \tag{1}$$

is readily seen to possess the two particular solutions

$$U_1 = xe^{-x\tanh t}\operatorname{sech}^2 t, \tag{2}$$

$$U_2 = e^{-x\coth t}. \tag{3}$$

These may be generalized by well-known methods so as to give solutions in the form of definite integrals

$$u = x\int_{-\infty}^{\infty} e^{-x\tanh z}\operatorname{sech}^2 z\, f(t-z)\, dz \tag{4}$$

$$u = \int_{0}^{\infty} e^{-x\coth z}\, g(t-z)\, dz, \tag{5}$$

in which $f(t)$ and $g(t)$ are suitable arbitrary functions. The first of these integrals can be expanded in the form of a power series

$$u = \sum_{n=0}^{\infty}{}_1 \frac{x^{n+1}}{n!}\, c_n(t) \tag{6}$$

in which

$$c_n(t) = \int_{-\infty}^{\infty} \tanh^n z \operatorname{sech}^2 z\, f(t-z)\, dz. \tag{7}$$

To obtain the general solution expressible as a power series of ascending powers of x we must add to (6) a constant multiple of the particular solution e^x.

The solution (4) admits the formal expansion

$$u = f(t)l_0(x) - f'(t)l_1(x) + f''(t)l_2(x) - \ldots \tag{8}$$

where

$$l_n(x) = \frac{x}{n!}\int_{-\infty}^{\infty} e^{-x\tanh z}\operatorname{sech}^2 z\, .\, z^n dz. \tag{9}$$

When x is real

$$|l_n(x)| < \frac{|x|e^{|x|}}{n!}\int_{-\infty}^{\infty} \operatorname{sech}^2 z\,|z|^n dz$$

$$= 8\,|x|e^{|x|}\left[\frac{1}{2^{n+1}} - 2\,\frac{1}{3^{n+1}} + 3\,\frac{1}{4^{n+1}} - \ldots\right]$$

$$< 8\,|x|e^{|x|}\,\frac{1}{2^{n+1}}. \qquad (10)$$

Hence if $f(t) = e^{\nu t}$ the series (8) certainly converges when $|\nu| < 2$ and represents a solution of type $u = e^{\nu t}F_\nu(x)$.

This function $F_\nu(x)$ may be found by first calculating the integral

$$\int_0^\infty e^{-\lambda x}\,F_\nu(x)dx \quad (\lambda > 1) \qquad (11)$$

which is found to be equal to

$$\frac{1}{\lambda^2-1}\left(\frac{\lambda+1}{\lambda-1}\right)^{\frac{\nu}{2}}\int_{-\infty}^{\infty}\frac{e^{-\nu s}\,ds}{\cosh^2 s}. \qquad (12)$$

We may therefore write (with the aid of a well-known inversion formula)

$$F_\nu(x) = \frac{1}{2\pi i}\int_C e^{\lambda x}\frac{d\lambda}{\lambda^2-1}\left(\frac{\lambda+1}{\lambda-1}\right)^{\frac{\nu}{2}}\int_{-\infty}^{\infty}\frac{e^{-\nu s}\,ds}{\cosh^2 s}, \qquad (13)$$

where, if $x > 0$, C is a contour which starts from $-\infty$ encircles the origin in the counter clockwise direction and returns to $-\infty$ so as to enclose the points -1 and $+1$ in a simple loop. Dropping the numerical factor depending only on ν we may easily verify that the contour integral

$$H_\nu(x) = \frac{1}{2\pi i}\int_C e^{\lambda x}(\lambda+1)^{\frac{\nu-2}{2}}(\lambda-1)^{-\frac{\nu+2}{2}}\,d\lambda \qquad (14)$$

satisfies the differential equation

$$x\frac{d^2H}{dx^2} = (x+\nu)H \qquad (15)$$

even when ν is not restricted by the inequality $|\nu| < 2$; it is, in fact, essentially the solution which Professor Kármán said he had found when he mentioned the differential equation to the author as one requiring further study.

It is easily seen that the function $H_\nu(x)$ satisfies the two recurrence relations

$$(\nu - 2)H_{\nu-2}(x) + (\nu + 2)H_{\nu+2}(x) = 2(\nu + 2x)H_\nu(x) \qquad (16)$$

$$(\nu + 2)H_{\nu+2}(x) - (\nu - 2)H_{\nu-2}(x) = 4x\, H'_\nu(x) \qquad (17)$$

which are very much like those satisfied by the function $k_\nu(x)$ discussed elsewhere.[1] Another function which satisfies the recurrence relations (16) and (17) is obtained by noticing that the definite integral (5) gives a solution of the form $e^{xt}h_\nu(x)$ where

$$h_\nu(x) = \int_0^\infty e^{-x\coth z - \nu z}\, dz. \qquad (18)$$

We have supposed in the derivation of these relations that $x > 0$, $\nu > 2$. The integral certainly has a meaning when $\nu = 2$ but is divergent when $\nu = 0$. To complete the definition of the function $h_\nu(x)$ we shall write for $x > 0$

$$h_0(x) = \int_0^\infty (e^{-x\coth z} - e^{-x})dz, \qquad (19)$$

observing that $h_0(x)$ is not a solution of the partial differential equation (1) but that a particular solution is given by

$$u = h_0(x) + t\, e^{-x}. \qquad (20)$$

An interesting addition formula for the function $h_0(x)$ is obtained by making use of the series

$$e^{-y\coth z} = k_0(y) - k_2(y)e^{-2z} + k_4(y)e^{-4z} - \ldots \qquad (21)$$

obtained in my former paper. With the aid of (18) and (19) we find that when $y > 0$, since $k_0(y) = e^{-y}$

$$S \equiv h_2(x)k_2(y) - h_4(x)k_4(y) + h_6(x)k_6(y) - \ldots$$
$$= \int_0^\infty e^{-x\coth z}\,[e^{-y} - e^{-y\coth z}]dz$$

the integration of the series term by term over the infinite range being justified by Dini's theorem (Bromwich's Infinite Series, 1st. ed., p. 455). Hence

$$S = e^{-y}\int_0^\infty (e^{-x\coth z} - e^{-x})dz - \int_0^\infty [e^{-(x+y)\coth z} - e^{-(x+y)}]dz$$
$$= k_0(y)h_0(x) - h_0(x + y)$$
$$\therefore \quad h_0(x + y) = k_0(y)h_0(x) - k_2(y)h_2(x) + k_4(y)h_4(x) - \ldots \qquad (22)$$

This is a particular case of a more general formula

$$h_\nu(x + y) = k_0(y)h_\nu(x) - k_2(y)h_{\nu+2}(x) + k_4(y)h_{\nu+4}(x) - \ldots \qquad (23)$$

which is easily proved with the aid of (18).

This relation may be used to prove the formula (s an integer)

$$\int_0^\infty h_\nu(x+y)k_{2s}(y)dy = (-)^s\left[h_{\nu+2s}(x) - \frac{1}{2}h_{\nu+2s-2}(x) - \frac{1}{2}h_{\nu+2s+2}(x)\right] \qquad s>0 \tag{24}$$

$$= \frac{1}{2}\left[h_\nu(x) - h_{\nu+2}(x)\right] \qquad s=0$$

A process of summation applied to the last formula gives

$$h_\nu(x) = \int_0^\infty m_{\nu-1}(x+y)k_0(y)dy \tag{25}$$

where

$$m_\nu(x) = \int_0^\infty e^{-x\coth z - \nu z}\,\operatorname{cosech} z.\,dz \tag{26}$$

$$k_0(y) = e^{-y} \tag{27}$$

$$x>0, \qquad y>0.$$

2. It should be mentioned that the differential equation (15) is a limiting case of the equation

$$4z\frac{d^2K}{dz^2} = 4(a-1)\frac{dK}{dz} + (z+2b)K \tag{28}$$

which Kummer[2] solves by means of a definite integral

$$K(a,b,z) = \int_0^{\frac{\pi}{2}}(\cos v)^{a-1}\cos\left(\frac{z}{2}\tan v + bv\right)dv \tag{29}$$

using a method which is only applicable when $a>1$. When $z=2x$, $a=1$, Kummer's function $K(a,b,z)$ becomes a constant multiple of the function which we have called $k_{-b}(x)$. In fact,

$$K(1,-\nu,2x) = \frac{\pi}{2}k_\nu(x). \tag{30}$$

Kummer expresses his function $K(a,b,z)$ in terms of the confluent hypergeometric function

$$\phi(\alpha,\gamma,z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots \tag{31}$$

which is now usually denoted by the symbol $F(\alpha,\gamma;z)$. The actual relation is

$$e^{\frac{z}{2}} K(a, b, z) = A\phi\left(\frac{b-a+1}{2}, b-a, x\right) + Bz^a \phi\left(\frac{b+a+1}{2}, 1+a, z\right)$$

where

$$A = \frac{\pi\Gamma(a)}{2a\,\Gamma\left(\frac{a-b+1}{2}\right)\Gamma\left(\frac{a-b+1}{2}\right)} \tag{32}$$

$$B = -\frac{\pi\cos\frac{a\pi-b\pi}{2}}{2a\,\Gamma(a+1)\sin(a\pi)}$$

This relation evidently fails to hold when $a = 1$.

The differential equation (15) is also a special case of an equation

$$x\frac{d^3y}{dx^3} = (a-1)\frac{d^2y}{dx^2} + (b+x)\frac{dy}{dx} + cy, \tag{33}$$

which Kummer finds to be the equation satisfied by the definite integral

$$y = \int_0^{\frac{\pi}{2}} (\sin\nu)^{c-1}(\cos\nu)^a \cos(x\tan\nu + b\nu)d\nu. \tag{34}$$

Equation (15) is, in fact, obtained by putting $a = 1$, $b = \nu$, $c = 0$, $H = dy/dx$. It should be noticed, moreover, that Kummer's formula

$$2\int_0^{\frac{\pi}{2}} (2\cos\nu)^m \cos[x\tan\nu + (m+2n)\nu]d\nu$$

$$= e^{\frac{x}{2}} \sin(n\pi) \int_0^1 u^m(1-u)^{n-1}e^{-\frac{2x}{u}}\,du \tag{35}$$

reduces, when $m = 0$, to the form

$$\pi k_{2n}(x) = 2\int_0^{\frac{\pi}{2}} \cos[x\tan\nu + 2n\nu]d\nu \tag{36}$$

$$= e^{\frac{x}{2}} \sin(n\pi) \int_0^1 (1-u)^{n-1} e^{-\frac{2x}{u}}\,du \quad (x > 0)$$

and furnishes an immediate proof of the theorem that $k_{2n}(x) = 0$ when n is a positive integer and $x > 0$.

It may be mentioned that Giuliani[3] has considered the two functions

$$U(a, b, x) = \int_0^{\frac{\pi}{2}} (\cos \nu)^{a-1} \cos (x \tan \nu) \cos b\nu \, d\nu$$
$$V(a, b, x) = \int_0^{\frac{\pi}{2}} (\cos \nu)^{a-1} \sin (x \tan \nu) \sin b\nu \, d\nu \qquad (37)$$

and finds that they satisfy one differential equation of the fourth order. An equation

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + (1 - x) \frac{\partial u}{\partial x} - u,$$

which is readily reducible to (1), has been discussed from a different standpoint by H. Faxén and has been used in connection with Ole Lamm's theory of diffusion and sedimentation in the ultracentrifugal separator.[4]

The solution (3) is readily deducible from M'Arthur's result[5] that the function

$$y = (1 - t)^{-p} \exp. \left(\frac{xt}{1 - t} \right)$$

is a particular solution of the partial differential equation

$$x \frac{\partial^2 y}{\partial x^2} + (p + x) \frac{\partial y}{\partial x} = t \frac{\partial y}{\partial t}.$$

The addition theorem (23) corresponds to the addition theorem

$$k_\nu (x + y) = k_\nu(x)k_0(y) + k_{\nu-2}(x)k_2(y) + k_{\nu-4}(x)k_4(y) + \ldots.$$
$$x > 0, y \geqq 0,$$

for the k-function.

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A SYNTHESIS OF THE THEOREMS OF HADAMARD AND HURWITZ ON COMPOSITION OF SINGULARITIES

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The two well-known classical theorems on composition of singularities due to Hadamard[1] and Hurwitz[2] one would expect to be interrelated in some way. In fact, Professor Hille has drawn my attention to this possibility in connection with my earlier paper on composition of singularities.[3]

The purpose of this paper is to prove the following theorem which contains, as special cases, the two theorems referred to above.

THEOREM. *Let the function* $q(x, u) = (au + bx + c)/(a'u + b'x + c')$ *depend both on* x *and* u. *Assume the following inequalities*

$$|b| > r_2\left(|b'| + \frac{|a'|r_0 + |c'|}{r}\right) + \frac{|a|r_0 + |c|}{r} \qquad (1)$$

$$(r_0 > r_1; \quad r > 0)$$

Let p *be zero or be a negative integer. Let the series* $\sum_{n=-1}^{\infty} a_n/x^{n+1}$, *with an isolated set of singularities* $\alpha_i[\,|\alpha_i| \leqq r_1 \ (\neq 0)]$, $\sum_{n=-1}^{\infty} b_n/x^{n+1}$, *with an isolated set of singularities* $\beta_i\,[\,|\beta_i| \leqq r_2 \ (\neq 0)]$ *represent the uniform functions* $f(x)$ *and* $g(x)$, *respectively. Moreover, let* C *denote a circle with center at the origin and with a radius* $R(r_1 < R < r_0)$.

It will follow then that the function

$$F(x) = \frac{1}{2\pi i}\int_C f(u)g(g(x, u)u^p du, \qquad (2)$$

is uniform and is representable, for $|x| > r$, *by the series*

$$\sum_{n=-1}^{\infty} C_n/x^{n+1}. \qquad (2a)$$

The singularities of $F(x)$ *will be all included among the expressions*

$$\frac{a - a'\beta_i}{b'\beta_i - b}\alpha_r + \frac{c - c'\beta_i}{b'\beta_i - b}, \qquad (3)$$

$$\frac{-(c'\alpha - a'c)}{(b'a - a'b)}, \qquad (3a)$$

$$\frac{\beta_k c' - c}{b - b'\beta_k} \text{ (if } p \leqq -i). \tag{3b}$$

There will be no singularities for $|x| > r$. *Furthermore, when* $c'b - b'c = 0$ *the expressions* (3a), (3b) *may be deleted.*

Hadamard's theorem will result when

$$b = a' = -p = 1, \quad a = b' = c = c' = 0,$$

while Hurwitz's theorem will result when

$$a_0 = b_0 = p = c = a' = b' = 0, \quad -a = b = c' = 1.$$

Proof.—Let u be within a Laurent ring $L(r_0)(0 < r_1 < |u| < r_0)$ and let x be in a region $R_0(|x| > r > 0)$. Then, writing $u' = q(x, u)$, we shall have

$$|u'| = \left|\left(b + \frac{au + c}{x}\right)\Big/\left(b' + \frac{a'u + c'}{x}\right)\right| > \frac{|b| - (|a| r_0 + |c|)/r}{|b'| + (|a'| r_0 + |c'|)/r}$$

provided that $|b| > (|a| r_0 + |c|)/r$. Now the latter inequality, as well as the inequality

$$|u'| > r_2 \qquad (u \text{ in } L(r_0);\ x \text{ in } R_0),$$

will be seen to hold in virtue of (1).

Consider now $F(x)$, as given by (2); C is obviously in $L(r_0)$. The series representing $f(u)$ and $g(u')$ are uniformly convergent in u for x in R_0, and it can be shown that $F(x)$ is analytic in R_0. Hence $F(x)$ will be represented by a series (2a). However, $F(x)$ is analytic in the more extensive region R_C, defined by the condition that none of the u-points given by

$$u = \alpha_i, \quad q(x, u) = \beta_k, \quad u = 0 \text{ (if } p \leqq -1) \tag{A}$$

shall lie within a distance $\delta(> 0)$ from C; here C is a contour not necessarily in $L(r_0)$. This is due to the fact that for u on C the integrand is analytic in x, for x in R_C, and that this integrand is a continuous function of x and u when x is in R_C and u is on C.

If C can be deformed into C_1 in a continuous way and without passing over any of the u-points of the set (A) we apparently have

$$\int_C f(u)g(q(x, u))u^p du = \int_{C_1} f(u)g(q(x, u))u^p du.$$

$F(x)$ exists in R_C. This region has with R_0 a neighborhood of $x = \infty$ in common. As x approaches the boundary of R_C some of the singular points in the u-plane may approach the boundary of C. We replace C

by C_1 as specified above. The integral along C_1 is analytic for x in R_{C_1}, that is, in a region including a neighborhood of $x = \infty$. Hence this integral is an analytic continuation of the series (2a). This process of continuation may fail only when points on opposite sides of C tend to coincide. Thus $F(x)$ is analytic (uniform) in every region such that for no point of the region do any two points of the set (A) coincide. The possible singularities of $F(x)$ are then given by (3), (3a), (3b).

Consider the situation when $c'b - b'c = 0$. Each of the expressions in (3a), (3b) will give $x = -c/b$. For this value of x we have $q(x, u) = a/a_1$. If now $a/a_1 \neq \beta i$ (for all i) we can see directly from (2) that $x = -c/b$ is not a singularity of $F(x)$. If, however, $a/a_1 = \beta_i$ (for some i) the value of x given by (3) (for this value of i) will be $(-c/b)$. This proves the assertion made in the Theorem to the effect that when $c'b - b'c = 0$ the expressions (3a), (3b) may be deleted.

It is a matter of direct verification to show that Hadamard's and Hurwitz's theorems come out as special cases as stated in the Theorem. It is essential to note that in these special cases the coefficients in (2a) will be desired functions of the a_n and the b_n. Hadamard's theorem will be seen to come out in a formulation different from, though equivalent to, the usual formulation.

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*REVERSIBLE COAGULATION IN LIVING TISSUE. VI**

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From experiments on the antagonistic action of sodium rhodanate to histamine with rabbits, Bancroft and Rutzler[1] deduced that histamine caused reversible coagulation of the proteins of the sympathetic nervous system. It is now proposed to show that histamine acts as a coagulating agent both on the proteins of the sympathetic nervous system and on those of the central nervous system.

Dale and Laidlaw[2] report that small doses of histamine produce narcosis in frogs, rabbits and cats, which means that in low concentrations histamine acts more strongly on the central nervous system than on the sympathetic nervous system. With the cat under the influence of a regular

anesthetic, "even moderate injections of histamine often produce a fatal circulatory collapse and respiratory failure." Here the anesthetic prevents the histamine from going into the brain and forces it to concentrate in the sympathetic nervous system with fatal results.

"The normal rabbit or guinea pig is easily killed by a relatively small intravenous injection of histamine but suffers little harm from a much larger injection when deeply under the influence of an anesthetic." In other words, a carnivorous animal behaves one way and a herbivorous animal another way. The explanation of this paradox is apparently to be found in the fact that the difference in effect between intravenous and subcutaneous injections is very marked with the rabbit and the guinea pig, and relatively slight with the cat. This must mean that the rabbit and the guinea pig detoxicate histamine fairly rapidly. If we assume that the anesthetic is taken up slightly by the sympathetic nervous system, its concentration may be low enough not to produce a shock itself and yet high enough so that the displacement by the histamine takes place too slowly to build up the shock concentration, even though intravenous injection is resorted to. Unfortunately, Dale and Laidlaw had no theory to guide them and consequently did not do experiments to clear up the matter.

When pointing out the likeness of histamine shock and anaphylactic shock, Dale and Laidlaw say that "all available evidence goes to show that the anaphylactic antibody is of the nature of a 'precipitin,' the interaction of which with the corresponding antigen results in a change in the state of dispersion of the colloidal particles." Wells[3] also draws attention to the close resemblance of histamine shock to anaphylactic shock. Kopaczewski[4] cites the work of Dale among others in support of his theory that shock is due to protein flocculation.

There can be no doubt but that the sympathetic nervous system takes part in the shock caused by histamine, for Chauchard and Saradjichvili[5] found in experiments with dogs that injections of histamine increased the chronaxie of the cardiac inhibitory fibers of the vago-sympathetic nerve. The chronaxie gradually returned to normal after the injections were stopped. "If the dose [of histamine] is excessive, the vago-sympathetic nerve cannot be excited and the animal soon dies." Bancroft and Richter[6] have shown that reversible protein agglomeration causes an increase in the chronaxie.

Histamine causes a loss of water in *Spirosteum teres*,[7] and the organism dies. The dead protozoon is much shrunken, and presents the appearance of having undergone heat coagulation. Bancroft and Rutzler[1] have shown that histamine dihydrochloride causes the coagulation of horse serum.

In order to study the effects of histamine on the sympathetic nervous system, excisions were made of the superior cervical ganglia of rabbits,

which is the purest sympathetic nervous tissue obtainable in bulk. In dealing with sympathetic nerves there is the advantage that the situation is not complicated by the presence of lipoids, for the sympathetic nerves have no medullary sheaths.[8]

The superior cervical ganglia of a good-sized rabbit were removed[9] after first pithing and bleeding the animal. The ganglia were removed as rapidly as possible after the animal was bled, and placed in a 15-cc. weighing-bottle containing Ringer's solution. These ganglia are so small (weighing between 14 and 56 milligrams) that a micro-balance was used. The ganglion was transferred from the Ringer's solution to a tightly-stoppered weighing-bottle by means of metallic forceps with fine ends; the ganglion was touched four or five times to the side of the bottle containing the Ringer's solution at a point above the surface of the liquid. The same thing was done in the weighing bottle and the "drip" subtracted from the apparent weight of the ganglion.

One ganglion was immersed in a $0.0604M$ histamine dihydrochloride solution and another in a $0.0604M$ sodium rhodanate solution. The one in the histamine solution shrank in size and the other swelled. When the ganglia were transposed, the change in weight reversed, showing that it was a case of reversible equilibrium.

Since ephedrine is a good peptizing agent[10] for proteins, it should follow that ephedrine would counteract histamine to some extent. Chen[11] reports that the fall in blood pressure in anesthetized dogs brought about by histamine, peptone and anaphylactic shock, is stopped, and the pressure raised, by the administration of ephedrine. Adrenaline[12] exerts a similar action. A one per cent solution of ephedrine hydrochloride arrests the heart in perfusion experiments; this arrest can be combated by two coagulating agents,[13] histamine and salts of calcium. This is, of course, a case of reversible peptization.

Special experiments on the skin reactions produced by histamine showed that these are checked somewhat by sodium rhodanate and ephedrine. By using a magnifying glass it was found possible to break the skin by means of a hypodermic needle to an approximately equal depth in each case.[14] A couple of drops of a solution containing 15 mg. histamine dihydrochloride per 100 cc. distilled water were applied carefully or a ten per cent solution of sodium rhodanate. The controls did not form wheals, and the erythema was negligible. With histamine alone the scratches itched; the wheals and erythrema lasted for nearly an hour. When the sodium rhodanate solution or Ringer's solution was applied before the histamine solution, there was no itching; the wheals and erythrema had practically stopped in about half an hour. Both Ringer's solution and rhodanate solution localize and reduce the skin reaction produced by histamine dihydrochloride.

Ephedrine, applied locally, should produce effects similar to those produced by sodium rhodanate. The left arm was cleaned properly and a ten per cent ephedrine sulphate solution was applied to one of the scratches. When this had dried, histamine solution was applied to both scratches. The histamine produced a strong reaction in the case of the untreated scratch; while the scratch that had been first treated with ephedrine showed hardly any reaction at all. This indicates that ephedrine is somewhat better than sodium rhodanate in its power to counteract the skin disturbance brought about by histamine. This was not altogether unexpected in view of the fact that ephedrine sulphate is a better peptizing agent for albumin sols than is sodium rhodanate.

If we are not dealing with reversible coagulation in these skin reactions, it is difficult to see why both sodium rhodanate and ephedrine should counteract the reactions as the theory predicted. It should be borne in mind that these two compounds, sodium rhodanate and ephedrine, nullify the action of histamine when that poison is injected into animal organisms, although sodium rhodanate tends to lower the blood pressure and ephedrine to raise it. This rules out changes in the blood pressure as the cause of the antagonisms.

The general conclusions of this paper are:

1. Histamine, a poison for the sympathetic nervous system, may also act upon the central nervous system, producing anesthesia under suitable conditions.

2. An anesthetic increases the action of histamine on carnivorous animals and decreases its action on herbivorous animals. With carnivorous animals the important thing seems to be the amount of histamine available. The anesthetic forces histamine from the brain to the sympathetic nervous system. With herbivorous animals the important thing appears to be the rate at which the histamine is supplied. The presence of some anesthetic in the sympathetic nervous system may be inadequate to produce shock; but may retard the rate at which adsorption of histamine takes place, so that there will be no shock. There is as yet no independent confirmation for this assumption in regard to the combined action on herbivorous animals.

3. The chronaxie of the vago-sympathetic nerve passes through a maximum with time after the injection of histamine. This is due to a reversible coagulation of the proteins.

4. Histamine coagulates horse serum and *Spirosteum teres*, as it should.

5. Swelling experiments show that the superior cervical ganglia (sympathetic nervous tissue) of rabbits are peptized by sodium rhodanate and agglomerated by histamine dihydrochloride.

6. The agglomeration of the proteins of the superior cervical ganglia of rabbits by histamine has been shown to be reversible.

7. *In vivo* both ephedrine and sodium rhodanate antagonize histamine, as one would expect from the fact that they are both good peptizing agents for proteins.

8. The skin reactions to histamine are checked by previous treatment with ephedrine or sodium rhodanate.

9. Reversible coagulation of proteins is probably responsible for the skin reactions, because histamine, calcium chloride, lactic acid and morphine—all agglomerating agents—cause skin reactions.

10. The evidence is overwhelming that the mode of action of histamine on the animal organisms is to produce reversible coagulation of the proteins.

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** Eli Lilly Research Fellow.

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[14] The authors thank H. P. Denniston, M.D., of Ithaca, who performed these experiments on the junior author.

REVERSIBLE COAGULATION IN LIVING TISSUE. VII*

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In an earlier paper[1] the erroneous conclusion was drawn that chloroform counteracted strychnine poisoning by acting on the same substrate. We now know that the injection of strychnine salts into animals lowers synaptic resistance in the spinal cord, which means serious impairment of normal inhibition. When an impulse from without comes into the spinal cord it has an uncontrolled spread to all the muscles of the body. This manifests itself as tetanic convulsions. The stimulus passes from the sensory nerves to the spinal cord and thence through the motor nerves to the muscles.

It is known that the tetanic convulsions of strychnine can be checked by administration of sodium rhodanate (a peptizing agent), ephedrine (a peptizing agent), chloroform (an agglomerating agent), curare (a motor nerve coagulant) and potassium salts (a muscle coagulant). We propose to show that the theory of Claude Bernard of the reversible coagulation of proteins enables us to account for the apparently similar action of such physiologically different substances.

Preliminary experiments showed that strychnine nitrate coagulates a turbid one percent egg-white sol, which proves that strychnine agglomerates animal proteins. As further confirmation several dorsal root ganglia of a rabbit were removed[2] and were tested with a 0.0604 molar strychnine nitrate solution. One ganglion lost five per cent of its weight in eight hours. Another ganglion gained twenty-six per cent in weight on immersion in a 0.0604 molar solution of sodium rhodanate, showing that this salt petptizes these proteins while the strychnine nitrate coagulates them. Strychnine nitrate had no visible effect on a lecithin sol, which makes it improbable that the lipoids play any vital part in tetanic convulsions.

Sodium chloride,[3] sodium bromide,[4] sodium iodide,[3] and sodium rhodanate[5] have been used to antagonize successfully the tetanic convulsions produced by strychnine. It is probable that the chief action of sodium rhodanate is to drive the strychnine out of the synapses and to bring these back toward a normal state. The halide salts probably act chiefly on the sensory nerves, as will be explained later. None of these salts are effective enough to warrant their employment in clinical practice. Dr. G. H. A. Clowes reports that ephedrine has a well-founded clinical use in combating strychnine poisoning, which is as it should be, because ephedrine is a better peptizing agent for egg white than sodium rhodanate is.

If anesthetics, which are coagulating agents, replaced strychnine on the same substrate, a high enough concentration of the anesthetic should

produce tetanic convulsions, which is apparently not the case. The anesthetic does not act through its effect on the brain, because decapitation[6] or decerebration does not necessarily stop the strychnine convulsions. On the other hand these convulsions are prevented[7] by cutting all of the sensory nerves leading to the spinal cord. An anesthetic coagulates the sensory nerves and thereby prevents the transmission of signals to the spinal cord. Cocainizing the entire skin will prevent strychnine spasms[8] even when the body is otherwise so sensitive that even a slight current of air is sufficient stimulus to precipitate tetanic convulsions. The message does not get through. Greene also says: "It is perfectly evident that sensory stimulation is necessary to the development of the tetanic contractions, but that the tetanus does not depend upon the toxic change in that mechanism."

Strychnine itself must cause some agglomeration of the sensory nerves because of the hypersusceptibility to stimuli, which appear to be equivalent to the excitability produced by anesthetics during the first stage of their action. Very large doses of strychnine do exert the sedative effect upon the peripheral nerves[9] that is to be expected in view of its anesthetic properties.

These considerations suggest that tetanic convulsions should be produced after injecting a sub-tetanic dose of strychnine into an animal, provided the sensory nerves are made more irritable by slight agglomeration in some other way. A rabbit weighing 2220 grams was given 2 cc. of a 0.3 per cent solution of strychnine nitrate by subcutaneous injection. Thirty-three minutes later 1.1 cc. more of the solution was given in the same way. Seventeen minutes later the rabbit became hypersensitive to touch and a few respiratory gasps were noted. At this time a cone wetted with chloroform was held loosely over the animal's head. A brief tetanic spasm occurred a few seconds after the anesthetic was started. The anesthetic was then administered rapidly to prevent any further convulsions. The experiment was repeated with the same result. The technique employed was such as to obviate the chance that impinging stimuli by the observer might bring on the tetanic convulsions.

While the sodium halides act to some extent on the same substrate as strychnine, the main part played by them is probably different. Sodium bromide, for instance, has a definite peptizing action on the sensory nerves. This may bring the sensory nerves back to normal, and the intensity of the stimulus transmitted may easily be small enough not to produce shock. We can thus reduce the hypersensitivity of the sensory nerves either by coagulating them more with chloroform or by peptizing them with sodium bromide.

The arrow poison, curare, changes the chronaxie of the motor nerves so that the messages do not go through from the spinal cord to the muscles. In more technical language, the efferent impulses through the sensory

nerves to the synapses in the spinal cord give rise to afferent impulses, which cannot get through the motor nerves to the muscles. Sollman[10] says that when curare is applied directly to the spinal cord of frogs, it causes typical *strychnine convulsions* [because it is a coagulating agent]. With ordinary method of administration these are masked by paralysis of the nerve endings.

If the muscles are paralyzed, no message can produce a result. Rabuteau recognized that potassium salts were muscle poisons and that it was on this account that they protected against strychnine. It seemed desirable to check this and accordingly a rabbit weighing 2080 grams was given an intravenous injection of 3 cc. of a 10 per cent solution of potassium chloride. Two minutes later, 2.3 cc. of a 0.3 per cent solution of strychnine nitrate was administered intravenously. This rabbit appeared to suffer no ill effects at all from this rather energetic treatment. The protection afforded by the potassium chloride was so unexpectedly complete that we distrusted our strychnine solution. To be on the safe side, a 2000-gram rabbit was given 2 cc. of the same strychnine solution intravenously. The rabbit went at once into severe tetanic convulsions and we saved its life by administering chloroform, thus killing two birds with one stone and without damaging the rabbit. It would be very desirable to know whether a potassium chloride solution would protect a man as well as it does a rabbit.

It is thus clear that the tetanic convulsions due to strychnine can be checked by displacing the strychnine from the synapses (sodium rhodanate or ephedrine), by anesthetizing the sensory nerves (chloroform), by peptizing the sensory nerves back to normal (sodium bromide), by changing the chronaxie of the motor nerves (curare), and by paralyzing the muscle so that they will not respond to the message when received (potassium chloride).

The action of strychnine as an anesthetic is interesting enough to justify a few extra words. Tetanic convulsions in themselves are not necessarily fatal. The deaths from strychnine are usually due to failure of the breathing apparatus. Since the frog can also breathe through his skin, strychnine is less likely to kill him and one can administer strychnine to get the anesthetic effect; and then the strychnine becomes temporarily an antidote to itself. The possibility of anesthetizing a dog with strychnine by giving him oxygen simultaneously is due to the breathing apparatus keeping going in oxygen when it breaks down in air.

The general conclusions of this paper are:

1. Strychnine causes convulsions by agglomerating certain proteins in the spinal cord, thereby doing away with certain inhibitions.

2. The dorsal root ganglia of the rabbit, which are affected by strychnine, are agglomerated when immersed in solutions of a strychnine salt.

3. Sodium rhodanate and ephedrine probably act by displacing the strychnine from the synapses. They will also act on the sensory nerves, as does sodium bromide.

4. Anesthetics stop strychnine convulsions by agglomerating the proteins of the sensory nerves, thereby preventing impulses from without from reaching the spinal cord.

5. Anesthetics do not counteract the tetanic convulsions of strychnine because they coagulate the brain colloids. A decerebrate or a decapitated frog can be made to have tetanic convulsions.

6. If a rabbit has been given a sub-tetanic dose of strychnine, a few whiffs of chloroform will thrown the rabbit into tetanic convulsions because the chloroform has increased the irritability of the sensory nerves.

7. Strychnine coagulates egg white. It causes anesthesia in frogs because the frog can breathe through the skin and consequently is less likely to die from the tetanic convulsions playing havoc with the respiratory organs. Giving a dog oxygen increases the efficiency of its respiratory organism and makes it possible for one to anesthetize a dog with strychnine.

8. Sodium bromide acts chiefly by peptizing the sensory nerves and restoring them to a more normal state. Ephedrine and sodium rhodanate must act similarly to some extent, though it is believed that their more important function is to displace strychnine from the synapses in the spinal cord.

9. Curare agglomerates egg-white sols and changes the chronaxie of the motor nerves so that no message gets through from the sensory nerves to the muscles. Consequently there are no tetanic convulsions.

10. Potassium chloride causes reversible agglomeration of the muscles. This is an action of the potassium ion.

11. Since potassium salts will coagulate the muscles, these are prevented from going into tetanic convulsions. With rabbits potassium chloride protects extraordinarily well against the tetanic convulsions due to strychnine.

12. Strychnine has no visible effect on a lecithin sol, which is an argument against the view that lipoids are a vital factor either in strychnine poisoning or in the protection against strychnine poisoning.

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** Eli Lilly Research Fellow.

[1] Bancroft and Rutzler, *J. Phys. Chem.*, **35**, 1185 (1931).

[2] Bancroft and Rutzler, *Ibid.*, **35**, 3036 (1931).

[3] Campo, *Physiol. Abs.*, **12**, 308.

[4] *U. S. Dispensatory*, 988 (1907).

[5] Bancroft and Rutzler, *Proc. Nat. Acad. Sci.*, **17**, 105 (1931).

[6] Rabuteau, *Éléments de Toxicologie*, 282 (1887).
[7] Hering, *Archiv exp. Path. Pharm.*, **38**, 276 (1896).
[8] Greene, *Handbook of Pharmacology*, 98 (1914).
[9] *U. S. Dispensatory*, 1194 (1907).
[10] Sollman, *Manual of Pharmacology*, 319 (1917).

RADIOACTIVE DISINTEGRATION

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There exist relationships between the energy of the γ-rays emitted by an atom following a β or α-particle emission and the fundamental physical constants.

The energy levels E of an optical electron can be represented, as it is well known,[1] by

$$-E/hc = \frac{RZ^2}{n^2}\left\{1 - {}^3/_4\frac{Z^2\alpha^2}{n^2} + \ldots \right\} \qquad (1)$$

where α is the fine structure constant and η an integer to a first approximation.

In the case of the energy levels E of the nucleus, we shall show that they can be represented by the following equation

$$E = -\left(\frac{n}{Z}\right)^2 \frac{\pi^2 Mc^2}{i^2}, \qquad (2)$$

where M = the mass of the helium nucleus
c = the velocity of light
Z = the atomic number of the element
n and i = integers.

For $Z = 92$, $n = 1$ and $i = 7$ we find for E the value -0.98×10^8 el. volts or for the corresponding wave length $\lambda = 136\ X.U.$

The agreement between this value for the wave length of the γ-rays, predicted from the equation (2) for the transition in which Δn^2 changes by one and the value of the wave length of the softest γ-rays emitted by the radioactive elements is shown in the following table.

From experiment we have for the softest rays emitted by

Rd-Th	UX_1	Pa	AcX
140 or 145	134	130	156

A second set has a wave length corresponding to

$i = 5$	Ra	ThB
	65	64

while a third set corresponds to $i = 9$

RaD	RaB	MsTh
261	230	212

The following elements satisfy the n^2 law fairly well, e.g.,

		AcX			
cal.	0.70	1.40	2.3	2.8	10^5 el. volts
exp.		(1.43)	2.0	2.7	
		(1.53)			
		(1.57)			
		ThB			
cal.	0.62	1.86	2.48	3.10	
exp.		1.91	(2.32)	3.02	
			(2.41)		
		RaB			
cal.	0.51	2.55	3.07	3.57	4.60
exp.	0.54	(2.43)	2.97	3.54	4.71
		(2.60)			

In the calculation of the energy of the γ-rays given in the last table, a value has been used for the energy level for $n = 1$ which fits the experimental data best instead of that predicted by equation (2). The difference in all these cases is only a few per cent as may be seen from the preceding table for the wave lengths of the softest γ-rays. This is in accord with the ideas of Rutherford and Ellis[2] who have shown in a recent paper that the energy of the γ-rays for RaB are integral multiples of a fundamental. They have also observed the same multiplet structure for RaB as is illustrated in the above example. The energy levels of RaB and ThB are double, while those of AcX are triple. This multiplicity of the levels seems to be connected with the number of protons P in the nucleus exclusive of those forming α particles or capable of forming α particles. That is, if W is the total number of protons in the nucleus, then $W - P = 4n$, where n is an integer and $P < 4$. The multiplicity is even or odd depending on whether P is even or odd. For the optical levels, the spin I of the nucleus seems to be related to the same factor by a relationship of a somewhat different character, e.g.,

$$I = P/2 + \text{an integer}.$$

A relationship of this kind has been proposed by Bartlett[3] to account for the variation of the nuclear spin with atomic weight.

The equation for the nuclear energy levels shows that the γ rays become softer the larger the value of Z. For elements for which $Z > 92$, the energy calculated by this equation becomes less in absolute value than the energy of the optical electron in the K shell. This fact may possibly be connected with the non-occurrence of elements of greater atomic number than 92.

In order to account for the stability of the nucleus, the electrons, protons and α-particles must be restricted to move in the vicinity of the nucleus by a potential barrier surrounding it.[4] The potential barrier will not be the same for electrons as for α-particles because the electron will be polarized differently by the electric field of the nucleus and because the electron possesses a magnetic moment. Further, the amount of energy in the nucleus requires that the size of the nucleus place an upper limit on the permitted values of "n" and on the number of "α" particles occupying any "n" level.

The mean life γ of a radioactive atom can also be correlated with the energy of the nucleus. The experimental data show that the reciprocal of the mean life $(1/\gamma)$ varies as the fourth power of the excess energy which the nucleus has after the emission of the α- or β-particle. By excess energy of the nucleus, we mean the energy of the α-particles in the nucleus over the energy in their ground state together with the energy of the β-particle. We shall take as a fair indication of the amount of this excess energy, the energy of the hardest γ-ray emitted following the β-particle emission. The constant of proportionality will depend on the position of the atom in the β-particle series. However, we shall first calculate the reciprocal of the mean life $(1/\gamma)$, using the constant of proportionality as calculated from the experimental data on UX_1 without distinguishing between the different groups.

β-RAY EMITTED AFTER α-PARTICLE

		$1/\gamma$ cal.	exp.
1st			
	RaB	2×10^{-4}	4×10^{-4}
	ThB	2.7×10^{-5}	1.8×10^{-5}
	UX_1	3.2×10^{-7}	3.2×10^{-7}
2nd			
	RaD	2×10^{-8}	1.3×10^{-9}
3rd			
	ThC''	2.4×10^{-1}	3.6×10^{-3}
	$MsTh_2$	3.2×10^{-3}	3.1×10^{-5}

The agreement for the elements in the first group is good, but for the elements in the second and third groups the calculated values are far too large. A much better agreement could be obtained if we were to choose the constant of proportionality for the second group to be 1/10 and that

of the third group 1/100 of its value in the first group. The constant of proportionality then changes from one group to another but has the same value in any one group as we should expect.

This relationship for β-particle emission is analogous to the emission of light quanta from a black body. The number of light quanta leaving the black body is proportional to the fourth power of the temperature; also the β-particle spectrum has an energy distribution similar to the continuous energy distribution in the black body radiation. This may mean that the properties of the "free" electrons in the nucleus can be calculated from thermodynamic considerations in a manner similar to that of light quanta in a hohlraum.

It is possible to write down a differential equation which has two sets of eigenwerte corresponding to those of equations (1) and (2). The solution will then give the eigenvalues for an electron either in the optical levels or in the nuclear ones; it will also give the energy values for an "α" particle in the nucleus by changing the value of the mass of the particle appearing in the equation from that of the electron to that of the particle and at the same time taking care of the difference in the electric charge in the two cases.

The radial component of the wave equation which will give these eigenwerte is

$$\left\{1 + \frac{A}{r} + \frac{B}{r^2}\right\}\left\{\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right\} + K\left\{E + \frac{zZe^2}{r} - \frac{l(l+1)}{r^2} + \frac{b}{r^3} + \frac{d}{r^4}\right\} R = 0$$

where $$K = \frac{8\pi^2 m}{h^2} \qquad A = \frac{iZh}{\sqrt{2\pi^2 M}}$$

and z takes account of the sign and magnitude of the charge on the particle. The solution is $R = e^{-i\sqrt{KE}\,r}\, r^{\nu}(1 + a_1 r + \ldots a_\mu r^\mu)$

with $$E = -\frac{Ke^4Z^2}{2(v+\mu+1)^2}\left\{1 - \tfrac{1}{2}\frac{Ke^2A}{(v+\mu+1)^2} + \ldots\right\}$$

or $$E = -\frac{4(v+\mu+1)^2}{KA^2},$$

where v and μ are integers.

An electron can assume either set of eigenwerte depending on its energy but an "α" particle can assume only the values corresponding to the second solution since the charge has the opposite sign to that of the electron. In the case of the "α" particle the first solution becomes infinite as $r = \infty$ and hence has no meaning for this particular problem.

The energy values for the electron in the nucleus are very much larger (in the ratio of their masses) than those of the "α" particle and hence will play no important rôle in the energy considerations of the nucleus except when they are attached to an "α" particle.[5] We also see that the larger the value of the energy the more restricted is the "α" particle to the region near the center of the nucleus. The "α" particles that fill the shells corresponding to large values of "n" or small values of Z will crowd toward the center and form a system similar to the electron system in the optical case.

The additional terms in the potential energy, b/r^3 and d/r^4, correspond to the energy arising from the polarizability of the particle in the field of the nucleus. The terms before the differential operator represent the change in mass of the electron arising from energy of interaction. The constant A is so chosen that the eigenwerte have the values demanded by the experimental data. This theory also leads to a value for the energy of the optical electron which is slightly different than that given by the Balmer formula. This difference is the same order of magnitude as that which appears in the hyperfine structure and to a first approximation the equation for the energy has the same form as regards the quantum numbers v and μ as the equation for the multiplet structure has for n and k. The optical electron can also suffer a perturbation arising from the interaction of the optical electron and the "α" particle in the nucleus, especially those with small values of n. This can be calculated from the known ψ functions of the optical electron and the "α" particle.

Although this paper is necessarily speculative in character, the model and the relationships derived from it, give a fairly accurate correlation of the data which may be valuable.

[1] Sommerfeld, *Wellenmechanischer Erganzungsband*, p. 336.

[2] Rutherford and Ellis, *Proc. Roy. Soc.*, **132**, p. 667.

[3] Bartlett, *Phys. Rev.*, Feb. 1, 1931.

[4] Gurney and Condon, *Phys. Rev.*, **33**, 137 (1929). Gamow, *Zeit. Phys.*, **51**, 204 (1928). Kudar, *Ibid.*, **53**, 61 (1929).

[5] NOTE: Since there is no evidence for γ-rays emitted by electrons, the electrons must be either in the ground state $n = 0$ or in stable states, e. g., in the structure of an "α" particle. The electrons in the ground state behave similar to the free electrons in a metal.

BODY TEMPERATURE AND HEAT REGULATION OF LARGE SNAKES

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In speaking of the "body temperature" of warm-blooded animals one is usually referring to the temperature of the mouth, the rectum or some one of the body cavities, such as the axilla or the groin. The snake may be considered as an elongated cylinder with no appendages and no cavities other than the mouth and the rectum. Since these are at a considerable distance from each other, it was thought that a study of the temperature in these two localities might give a hint as to the temperature distribution, although the technical difficulties of securing the correct body temperature of the snake are great. In table 1 is given a comparison of temperatures taken in the rectum and in the mouth of a large python.

Contrary to the situation obtaining with warm-blooded animals, where the temperature in the rectum is invariably higher than that in any other part of the body, this comparison shows that with the python the temperature is essentially the same in the mouth and in the rectum. A comparison

TABLE 1

Comparison of Mouth and Rectal Temperatures of a Python

DATE	TIME	BODY TEMPERATURE, °C	
		RECTAL	MOUTH
Dec. 20, 1916	1.45 P.M.	29.0	29.2
Dec. 20, 1916	1.50 P.M.	28.9	....
Jan. 17, 1917	3.00 P.M.	25.9	25.3
Jan. 17, 1917	4.20 P.M.	25.9	26.1
Jan. 18, 1917	11.05 A.M.	23.8	23.7
Jan. 18, 1917	4.30 P.M.	21.9	22.6
Mar. 2, 1917	11.19 A.M.	26.9	26.4
Mar. 2, 1917	11.31 A.M.	24.3	24.4

was likewise made of the rectal temperature and the temperature of the skin in contact with the air. By use of a thermo-junction method, skin temperature measurements were obtained on a large snake every 10 to 15 cm. down the entire length of the animal. When these were compared with temperatures taken in the rectum, it was found that the temperature of the skin was, as a rule, somewhat lower than that of the rectum. Thus, when the rectal temperature was 29.55°C., the skin temperature averaged 28.91°C., and in another instance when the rectal temperature was 22.57°C. the average skin temperature was 21.74°C.

The extraordinary reaction of these cold-blooded animals to changes in environmental temperature may best be shown by a comparison of environmental and rectal temperatures, when considerable and rapid changes in environmental temperature were artificially produced. Curves obtained with a fasting boa under such conditions, curves typical of a number of similar experiments, are shown in figure 1. In these cases the rectal temperature, shown by the solid black line in figure 1, was invariably lower than the environmental temperature, save when the environmental temperature was suddenly dropped. Then the rectal temperature rather

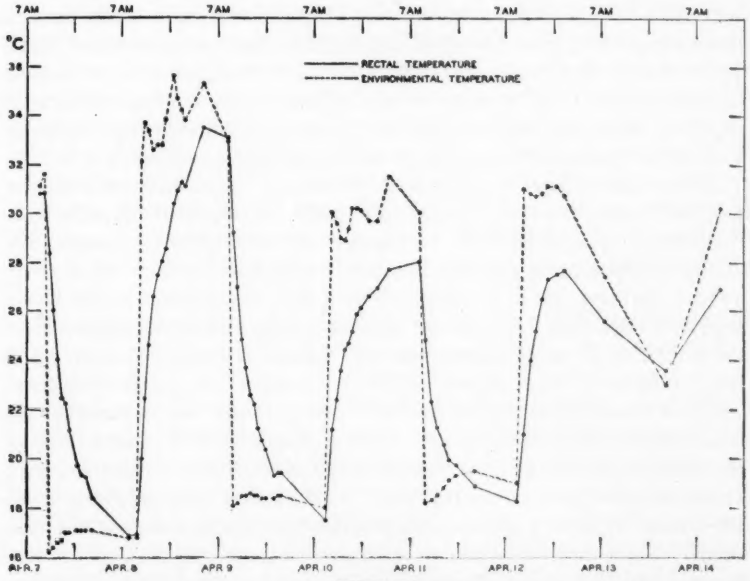


FIGURE 1

Influence of changes in environmental temperature upon the rectal temperature of the snake.

slowly assumed a lower level but ultimately passed below the environmental temperature, even at 18°C. The evidence demonstrates that the rectal temperature of the snake during fasting is practically always below its environmental temperature.

As a result of muscular activity the snake's rectal temperature may actually exceed and, indeed, rise considerably above the environmental temperature. Thus, when snakes were placed upon the floor and purposely agitated so that they made rapid and angry attempts to strike the experimenter, their rectal temperatures were invariably above the environmental

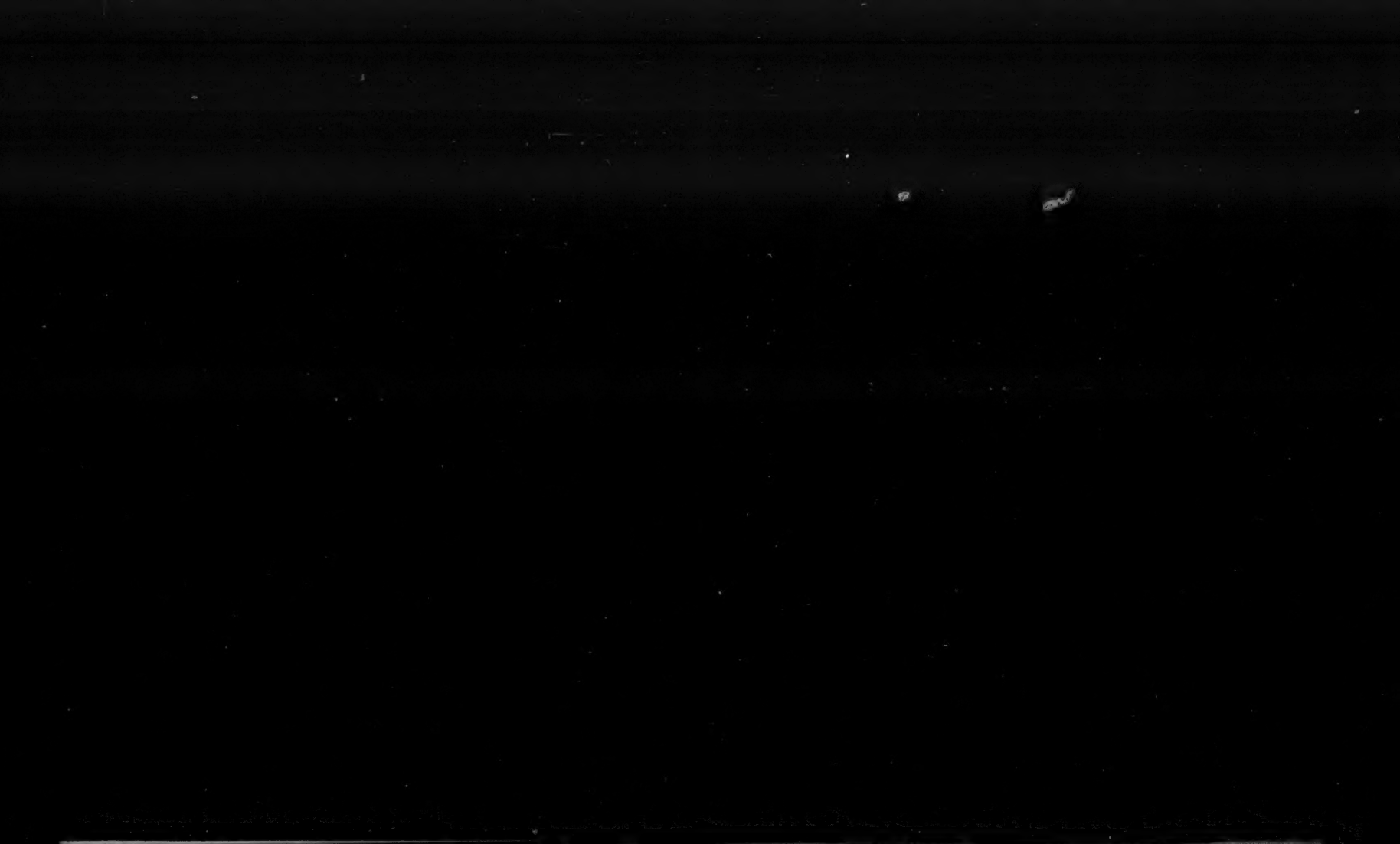
temperature. One would expect that the increased heat production resulting from muscular activity might not be immediately compensated by an increased loss of heat, and that for this reason there would be an increased body temperature. Is it possible to produce by any means other than by muscular activity, by some internal process, for example, a condition in which the heat loss cannot keep pace with the heat production and the rectal temperature will therefore be above the environmental temperature? Observations made on snakes at the peak of active digestion showed that the rectal temperature was definitely higher than the environmental temperature, indeed several degrees above it. When the thermojunction was passed over the obviously distended portion of the snake's body where the ingested food was undergoing digestion, it was also found that the skin temperature was slightly higher on this part of the body than on other parts. Thus in this cold-blooded animal the internal processes of digestion, as well as muscular activity, produce an internal body temperature higher than the temperature of the animal's surroundings.*

These observations, namely, that the fasting snake has a rectal temperature higher than the temperature of the environment, that its body temperature fluctuates with the changes in environmental temperature and is increased above the environmental temperature by muscular activity and by digestion, led to a consideration of the various paths for the loss of heat with these large cold-blooded animals. In general about 25 per cent of the total heat given off by warm-blooded animals is eliminated as the latent heat of water of vaporization. From our temperature study with large snakes it seemed clear that there could be little, if any, loss of sensible heat from these cold-blooded animals. Obviously sensible heat could be lost by the snake, if its body temperature, as a result of activity or of digestion, were above the environmental temperature. A calorimeter was available in which the animal could be placed and the sensible heat, if any, detected and measured. These direct calorimetric measurements, made by V. Coropatchinsky, showed that under all conditions of repose and not during digestive activity the snake gave off no sensible heat, and that all the heat resulting from the metabolism was lost from the animal through the vaporization of water. Inasmuch as an electrical compensation calorimeter was used, it became necessary to introduce heat by the electric current into the chamber with the animal, in order to maintain a constant temperature inside the chamber and to compensate for the heat carried away by the vaporization of water. In view of these calorimetric studies the rôle of water

* Since this report was prepared for publication an extensive series of observations have been made of the skin temperature of an incubating python, as compared with the environmental temperature. This survey demonstrates conclusively that the body temperature of the python is definitely higher than the temperature of its environment during the period of incubation.

in the metabolism of the snake assumes a new significance. The snake is reported to have no sweat glands. The water vapor lost from its body must therefore escape in the expired air and, by process of diffusion and evaporation, through the skin. Since apparently all the heat produced by these animals under conditions of post-digestive muscular repose is lost through the vaporization of water, it is obvious that material amounts of water must be thus given off.

In a number of instances the amount of water vaporized was determined directly by placing the snake inside a respiration chamber and absorbing in a suitable reagent the water vapor present in the air current coming from the chamber. The gaseous metabolism (carbon-dioxide production and oxygen absorption) was likewise determined. The water vaporized amounted to as much as 4 or 5 grams per kilogram of body weight per 24 hours, when the environmental or rectal temperature was around 30°C. It was increased profoundly by the previous handling and agitation of the animal and was increased likewise by digestion. Similarly, the insensible perspiration, that is, the loss in body weight from hour to hour when feces and urine were not passed, was noticeable. The animal was weighed in a cage or box suspended from the arm of a balance. It was found to lose weight continually, this loss in weight being roughly approximate to the measured water vapor.

The insensible perspiration could not well be determined simultaneously with the direct determination of the water vapor, but at all events it is certain that with the snake, notwithstanding the absence of sweat glands and its low metabolism, a considerable amount of water is vaporized. Since the body temperature of the snake is for the most part below that of the environment and its skin temperature is much below, in contradistinction to the situation with warm-blooded animals, one may consider that in general the snake partakes somewhat of the nature of a wet bulb thermometer. The vaporization of water from the skin and the lungs definitely lowers the temperature of the body. Indeed, it has been proved that the amount of water vapor lost from the snake at a high temperature is much more than that which would be required simply to take care of the heat produced by the animal. Hence there is undoubtedly absorption of heat by the animal from the environment, with correspondingly an increased water of vaporization. With these cold-blooded animals, therefore, the vaporization of water represents the main path for heat loss, a path that with warm-blooded animals takes care of only about one-fifth of the total heat loss.

The details of these and other experiments with snakes, together with a full account of observations carried out on large tortoises, lizards and alligators, are shortly to be published in monograph form by the Carnegie Institution of Washington.